

Point-curve incidences in the complex plane

Adam Sheffer*

Endre Szabó†

Joshua Zahl‡

June 27, 2016

Abstract

We prove an incidence theorem for points and curves in the complex plane. Given a set of m points in \mathbb{R}^2 and a set of n curves with k degrees of freedom, Pach and Sharir proved that the number of point-curve incidences is $O(m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} + m + n)$. We establish the slightly weaker bound $O_\varepsilon(m^{\frac{k}{2k-1} + \varepsilon} n^{\frac{2k-2}{2k-1}} + m + n)$ on the number of incidences between m points and n (complex) algebraic curves in \mathbb{C}^2 with k degrees of freedom. We combine tools from algebraic geometry and differential geometry to prove a key technical lemma that controls the number of complex curves that can be contained inside a real hypersurface. This lemma may be of independent interest to other researchers proving incidence theorems over \mathbb{C} .

1 Introduction

Let \mathcal{P} be a set of points and let \mathcal{V} be a set of geometric objects (for example, one might consider lines, circles, or planes) in a vector space K^d over a field K . An *incidence* is a pair $(p, V) \in \mathcal{P} \times \mathcal{V}$ such that the point p is contained in the object V . In incidence problems, one is usually interested in the maximum number of incidences in $\mathcal{P} \times \mathcal{V}$, taken over all possible sets \mathcal{P}, \mathcal{V} of a given size. For example, the well-known Szemerédi-Trotter Theorem [25] states that any set of m points and n lines in \mathbb{R}^2 must have $O(m^{2/3}n^{2/3} + m + n)$ incidences.

Incidence theorems have a large variety of applications. For example, in the last few years they have been used by Guth and Katz [12] to almost completely settle Erdős' distinct distances problem in the plane; by Bourgain and Demeter [3, 2] to study restriction problems in harmonic analysis; by Raz, Sharir, and Solymosi [21] to study expanding polynomials; and by Farber, Ray, and Smorodinsky [10] to study properties of totally positive matrices.

1.1 Previous work

We will be concerned with the number of incidences between points and various classes of curves. Later, we will define several different types of curves, but for the definition below one can think of a curve as merely a subset of K^2 , where K is either \mathbb{R} or \mathbb{C} .

Let \mathcal{C} be a set of curves in K^2 and let \mathcal{P} be a set of points in K^2 . We say that the arrangement $(\mathcal{P}, \mathcal{C})$ has k degrees of freedom and multiplicity type s if

- For any subset $\mathcal{P}' \subset \mathcal{P}$ of size k , there are at most s curves from \mathcal{C} that contain \mathcal{P}' .
- Any pair of curves from \mathcal{C} intersect in at most s points from \mathcal{P} .

*California Institute of Technology, Pasadena, CA, adamsh@gmail.com.

†Alfréd Rényi Institute of Mathematics, Budapest, szabo.endre@renyi.mta.hu.

‡University of British Columbia, Vancouver, BC, jzahl@math.ubc.ca.

We will use $I(\mathcal{P}, \mathcal{C})$ to denote the number of incidences between the points in \mathcal{P} and curves in \mathcal{C} . The current best bound for incidences between points and general curves in \mathbb{R}^2 is the following (better bounds are known for some specific types of curves, such as circles and parabolas¹).

Theorem 1.1 (Pach and Sharir [19]). *Let \mathcal{P} be a set of m points in \mathbb{R}^2 and let \mathcal{C} be a set of n simple plane curves. Suppose that $(\mathcal{P}, \mathcal{C})$ has k degrees of freedom and multiplicity type s . Then*

$$I(\mathcal{P}, \mathcal{C}) = O_{k,s} \left(m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} + m + n \right).$$

If the curves are algebraic, then we can drop the requirement that the curves are simple (however, the implicit constant will now depend on the degree of the curves). This special case was proved several years earlier than Theorem 1.1. The proof follows from the techniques in [5], and appears explicitly in [18].

Theorem 1.2 (Pach and Sharir [18, 19]). *Let \mathcal{P} be a set of m points in \mathbb{R}^2 and let \mathcal{C} be a set of n algebraic curves of degree at most D . Suppose that $(\mathcal{P}, \mathcal{C})$ has k degrees of freedom and multiplicity type s . Then*

$$I(\mathcal{P}, \mathcal{C}) = O_{k,s,D} \left(m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}} + m + n \right). \quad (1)$$

Less is known about point-curve incidences in the complex plane. If we add the additional requirement that pairs of curves must intersect transversely², then an analogue of Theorem 1.2 can be proved using the techniques of Solymosi-Tao from [24], although these methods introduce an ε loss in the exponent. Previously, Tóth [26] proved the important special case where the curves in \mathcal{C} are lines. This was generalized by the third author in [28], who proved a bound analogous to that in Theorem 1.2 for complex curves. However, in addition to the requirement that curves intersect transversely, the results of [28] have an additional restriction on the relative sizes of \mathcal{P} and \mathcal{C} , and they require that the curves be smooth. Elekes and the second author [9, Theorem 9] proves Pach-Sharir-like estimates for arbitrary complex subvarieties in \mathbb{C}^d , but their exponent is far from optimal. Finally, Dvir and Gopi [8] and the third author [29] consider incidences between points and lines in \mathbb{C}^d , for any $d \geq 3$.

Asking for the curves to intersect transversely is rather restrictive; some of the simplest cases such as incidences with circles or parabolas do not satisfy this requirement. If we do not require that pairs of curves intersect transversely, then much less is known. Very recently, Solymosi and de Zeeuw [23] proved a complex analog of Theorem 1.2, but only for the special case where the point set is a Cartesian product $A \times B \subset \mathbb{C}$. This bound has already been used to prove several results—see [20, 27].

1.2 New results

We obtain a complex analogue of Theorem 1.2, although our version introduces an ε loss in the exponent.

Theorem 1.3. *For each $k \geq 1, D \geq 1, s \geq 1$, and $\epsilon > 0$, there is a constant $C = C_{\epsilon,D,s,k}$ so that the following holds. Let $\mathcal{P} \subset \mathbb{C}^2$ be a set of m points and let \mathcal{C} be a set of n complex algebraic curves of degree at most D . Suppose that $(\mathcal{P}, \mathcal{C})$ has k degrees of freedom and multiplicity type s . Then*

$$I(\mathcal{P}, \mathcal{C}) \leq C \left(m^{\frac{k}{2k-1} + \epsilon} n^{\frac{2k-2}{2k-1}} + m + n \right). \quad (2)$$

¹Recently, Sharir and the third author obtained an improvement [22] to Theorem 1.1 whenever the curves are algebraic.

²That is, whenever two complex curves intersect at a smooth point of both curves, their complex tangent lines at the point of intersection are distinct.

The new improvement is that Theorem 1.3 does not require the curves to intersect transversely. To do this, we need several new ideas which are discussed below.

1.3 Proof sketch

Each point of \mathcal{P} can be regarded as a point in \mathbb{R}^4 , and each curve of \mathcal{C} can be regarded as a two-dimensional variety in \mathbb{R}^4 . Thus the problem is reduced to bounding the number of incidences between a set of points in \mathbb{R}^4 and a set \mathcal{S} of two-dimensional surfaces in \mathbb{R}^4 . If every pair (or at least most pairs) of surfaces in \mathcal{S} intersect transversely, then the bound (2) can be obtained by using the techniques of Solymosi and Tao from [24]. However, if many pairs of surfaces in \mathcal{S} fail to intersect transversely, then the techniques from [24] do not apply.

Luckily, the surfaces in \mathcal{S} are special—they come from complex curves in \mathbb{C}^2 . More precisely, the surfaces in \mathcal{S} are the images of complex curves in \mathbb{C}^2 under the usual embedding of \mathbb{C}^2 into \mathbb{R}^4 . In particular, the surfaces in \mathcal{S} are defined by pairs of real polynomials that satisfy the Cauchy-Riemann equations. As we will show below, this means that the only way that many surfaces in \mathcal{S} can lie in a common low-degree hypersurface Z is if the surfaces are leaves of a foliation of Z . Of course, if the surfaces are leaves of a foliation then they are disjoint, so the total number of point-surface incidences is small. Section 3 is devoted to making this statement precise. One of our main technical results is Theorem 3.3 and Corollary 3.4. This result says that for most points z of a hypersurface $Z \subset \mathbb{R}^4$, there can be at most one (embedded) complex curve from \mathcal{C}^2 that is contained in Z and incident to z .

To prove Theorem 3.3, we will associate with most points $p \in Z$ a two dimensional plane H_p that is contained in the tangent space $T_p Z$. We then show that if $\gamma \subset \mathbb{C}^2$ is a complex curve whose image in \mathbb{R}^4 contains p and is contained in Z , then the tangent space of (the image of) γ at p must be equal to H_p . Thus, the embedding of every complex curve has the same tangent plane at p . The planes $\{H_p\}$ define a sub-bundle of the tangent bundle TZ . We show that this sub-bundle is involutive (or integrable), and this means that Z can be foliated into a disjoint union of two-dimensional leaves. If the embedding of an irreducible complex plane curve is contained in Z , then this embedded complex curve must be a leaf (or several leaves) of this foliation. In particular, at most one embedded curve can pass through a typical point of Z . This is the key technical innovation that allows us to obtain the desired bounds for Theorem 1.3.

2 Preliminaries

2.1 Varieties and ideals

In this paper we work over the fields \mathbb{R} and \mathbb{C} . Let $K = \mathbb{R}$ or \mathbb{C} . Varieties are (possibly reducible) Zariski closed subsets of K^d . If $X \subset K^d$ is a set, let \overline{X} be the Zariski closure of X ; this is the smallest variety in K^d that contains X .

If $Z \subset \mathbb{R}^d$ is a variety, let $Z^* \subset \mathbb{C}^d$ be the smallest complex variety containing Z ; i.e., Z^* is obtained by embedding Z into \mathbb{C}^d and then taking the Zariski closure. If $Z \subset \mathbb{C}^d$, let $Z(\mathbb{R}) \subset \mathbb{R}^d$ be the set of real points of Z . We also identify \mathbb{C}^2 with \mathbb{R}^4 using the map $\iota(x_1 + iy_1, x_2 + iy_2) = (x_1, y_1, x_2, y_2)$ (where $x_1, y_1, x_2, y_2 \in \mathbb{R}$). If \mathcal{C} is a set of curves in \mathbb{C}^2 , we define $\iota(\mathcal{C}) = \{\iota(\gamma) : \gamma \in \mathcal{C}\}$.

If $Z \subset K^d$ is a variety, let $I(Z)$ be the ideal of polynomials in $K[x_1, \dots, x_d]$ that vanish on Z . If $I \subset K[x_1, \dots, x_d]$ is an ideal, let $\mathbf{Z}(I) \subset K^d$ be the intersection of the zero-sets of all polynomials in I . Sometimes it will be ambiguous whether an ideal is a subset of $\mathbb{R}[x_1, \dots, x_d]$ or $\mathbb{C}[x_1, \dots, x_d]$. To help resolve this ambiguity, we will write $\mathbf{Z}_{\mathbb{R}}(I)$ or $\mathbf{Z}_{\mathbb{C}}(I)$. If $P \in K[x_1, \dots, x_d]$ is a polynomial,

we abuse notation and write $\mathbf{Z}(P)$ instead of $\mathbf{Z}((P))$. If $I \subset \mathbb{C}[x_1, \dots, x_d]$ is an ideal, we use $\sqrt{I} = I(\mathbf{Z}(I))$ to denote the radical of I .

Often in our arguments we will refer to properties that hold for most points on a variety. To make this precise, we will introduce the notion of a generic point. Let $Z \subset \mathbb{C}^d$ be an irreducible variety, and let M be a finite set of polynomials, none of which vanish on Z . We say that a point $z \in Z$ is *generic* (with respect to M) if none of the polynomials in M vanish at z . In particular, for Z and M fixed, the set of generic points of Z is Zariski dense in Z .

In practice, the set of polynomials will be apparent from context, so we will abuse notation and simply refer to generic points. In general, the set of polynomials M will depend on the variety Z , the points and curves from the statement of Theorem 1.3, any previously defined objects, and whatever property is currently under consideration.

If $Z(\mathbb{R})$ is Zariski dense in Z , then we define a generic real point of $Z(\mathbb{R})$ to be a point $z \in Z(\mathbb{R})$ for which no polynomial in M vanishes. In particular, if $Z(\mathbb{R})$ is dense in Z , then Z always contains a generic real point.

Finally, we will sometimes refer to generic linear spaces or generic linear transformations. A generic linear space of dimension ℓ in \mathbb{C}^d is a generic point of the Grassmannian of ℓ -dimensional vector spaces in \mathbb{C}^d . Similarly, a generic linear transformation in \mathbb{C}^d is a generic point of $\mathrm{GL}(\mathbb{C}, d)$.

The degree of an irreducible affine variety $V \subset \mathbb{C}^d$ of dimension d' is the number of points of the intersection of V with a generic linear space of dimension $d - d'$ (for several equivalent definitions, see [13, Chapter 18]). We define the degree of a reducible variety V as the sum of the degrees of the irreducible components of V (note that these components may have different dimension). In practice, we are only interested in showing that the degrees of various varieties are bounded, so the specific definition of degree is not too important.

Lemma 2.1 (Varieties and their defining ideals). *Let $Z \subset \mathbb{C}^d$ be a variety of degree C . Then there exist polynomials f_1, \dots, f_ℓ such that $(f_1, \dots, f_\ell) = I(Z)$ and $\sum_{j=1}^\ell \deg f_j = O_{C,d}(1)$.*

Proof. This is essentially [4, Theorem A.3]. In [4], the authors prove the weaker statement that there exists a set of polynomials g_1, \dots, g_t such that $\sum \deg g_j = O_{d,C}(1)$ and $I(Z) = \sqrt{(g_1, \dots, g_t)}$. However, a set of generators for $\sqrt{(g_1, \dots, g_t)}$ can then be computed using Gröbner bases (see e.g. [6] for an introduction to Gröbner bases). The key result is due to Dubé [7], which says that a reduced Gröbner basis for (g_1, \dots, g_t) can be found (for any monomial ordering) such that the sum of the degrees of the polynomials in the basis is $O_{d,C'}(1)$, where $C' = \sum \deg g_j$. Since $C' = O_{d,C}(1)$, we conclude that the sum of the degrees of the polynomials in the Gröbner basis is $O_{d,C}(1)$. Once a Gröbner basis for (g_1, \dots, g_t) has been obtained, a set of generators for $\sqrt{(g_1, \dots, g_t)}$ can then be computed (see e.g. [11, Section 9]). \square

2.2 Regular points, singular points, and smooth points

We will often refer to the *dimension* of an affine real algebraic variety. Informally, a real algebraic variety X has dimension d' if there exists a subset of X that is homeomorphic to the open d' -dimensional cube, but there does not exist a subset of X that is homeomorphic to the open $(d' + 1)$ -dimensional cube. See [1] for a precise definition of the dimension of a real algebraic variety.

Let $X \subset \mathbb{R}^d$ be a variety of dimension d' and let $\zeta \in X$. We say that ζ is a *smooth* point of X if there is a Euclidean neighborhood $U \subset \mathbb{R}^d$ containing ζ such that $X \cap U$ is a d' -dimensional embedded submanifold; for example, see [1, Section 3.3]. In this paper we only consider smooth manifolds, and for brevity we refer to these simply as manifolds. Let X_{smooth} be the set of smooth points of X ; then X_{smooth} is a d' -dimensional smooth manifold.

Similarly, let $X \subset \mathbb{C}^d$ be a variety of dimension d' and let $\zeta \in X$. We say that ζ is a *smooth* point of X if there is a Euclidean neighborhood $U \subset \mathbb{C}^d$ containing ζ such that $X \cap U$ is a d' -dimensional embedded complex submanifold. Again, let X_{smooth} be the set of smooth point of X ; then X_{smooth} is a d' -dimensional complex manifold.

Let $X \subset \mathbb{C}^d$ be a variety of pure dimension d' (i.e., all irreducible components of X have dimension d'), and let f_1, \dots, f_ℓ be polynomials that generate $I(X)$. We say that $\zeta \in X$ is a *regular* point of X if

$$\text{rank} \begin{bmatrix} \nabla f_1(\zeta) \\ \vdots \\ \nabla f_\ell(\zeta) \end{bmatrix} = d - d'. \quad (3)$$

Let X_{reg} be the set of regular points of X . If $\zeta \in X$ is not a regular point of X , then ζ is a *singular* point of X . Let X_{sing} be the set of singular points of X .

Lemma 2.2 ([17], Corollary 1.26). *Let $X \subset \mathbb{C}^d$ be a variety of pure dimension d' . Then $X_{\text{smooth}} = X_{\text{reg}}$.*

Lemma 2.3. *Let $X \subset \mathbb{C}^d$ be a variety of degree C . Then X_{sing} is a variety of dimension strictly smaller than $\dim(X)$, and $\deg(X_{\text{sing}}) = O_{C,d}(1)$.*

Proof. By Lemma 2.1, there exist polynomials f_1, \dots, f_ℓ such that $(f_1, \dots, f_\ell) = I(X)$ and $\sum_{j=1}^\ell \deg f_j = O_{C,d}(1)$. We have

$$X_{\text{sing}} = \left\{ \zeta \in X : \text{rank} \begin{bmatrix} \nabla f_1(\zeta) \\ \vdots \\ \nabla f_\ell(\zeta) \end{bmatrix} < d - d' \right\}. \quad (4)$$

Equation (4) shows that X_{sing} can be written as the zero locus of $O_\ell(1) = O_{d,C}(1)$ polynomials, each of degree $O_{d,C}(1)$. Thus X_{sing} is a variety of degree $O_{d,C}(1)$. It remains to prove that X_{sing} has dimension strictly smaller than $\dim(X)$. This property can be found, for example, in [14, Chapter I, Theorem 5.3]. \square

2.3 The Cauchy-Riemann equations

Let $K = \mathbb{R}$ or \mathbb{C} . Let $u, v: K^4 \rightarrow K$ be functions. We will write $u = u(x_1, y_1, x_2, y_2)$, $v = v(x_1, y_1, x_2, y_2)$. If $K = \mathbb{R}$, then we require that $u, v \in C^2(\mathbb{R}^4)$. If $K = \mathbb{C}$, then we require that u and v be holomorphic. We say that the pair (u, v) satisfies the Cauchy-Riemann equations at the point (x_1, y_1, x_2, y_2) if

$$\frac{\partial u}{\partial x_k} = \frac{\partial v}{\partial y_k}, \quad \frac{\partial u}{\partial y_k} = -\frac{\partial v}{\partial x_k}, \quad k = 1, 2. \quad (5)$$

Here the partial derivatives are either real derivatives (if $K = \mathbb{R}$ and u, v are in $C^2(\mathbb{R}^4)$), or complex derivatives (if $K = \mathbb{C}$ and u, v are holomorphic). If (5) holds at every point of K^4 , then we say that the pair (u, v) satisfies the Cauchy-Riemann equations.

One important property of the Cauchy-Riemann equations is that when $K = \mathbb{R}$ and u and v are in $C^2(\mathbb{R}^4)$, then (u, v) satisfies the Cauchy-Riemann equations if and only if $f = u + iv$ is holomorphic.

Observe that if (u, v) satisfy the Cauchy-Riemann equations at the point $\zeta = (x_1, y_1, x_2, y_2)$, and if $\nabla u(\zeta) = (a_1, a_2, a_3, a_4)$, then $\nabla v(\zeta) = (-a_2, a_1, -a_4, a_3)$. If u and v take real values, then $\nabla u(\zeta)$ and $\nabla v(\zeta)$ must be orthogonal (provided $\nabla u(\zeta) \neq 0$), and in particular, $\nabla u(\zeta)$ and $\nabla v(\zeta)$

are linearly independent. If u and v are complex-valued, however, then this need not be the case. For example, if $a_2 = ia_1$ and $a_4 = ia_3$, then $(a_1, a_2, a_3, a_4) = i(-a_2, a_1, -a_4, a_3)$. Luckily, however, this example is the only one.

Lemma 2.4. *Let $a_1, a_2, a_3, a_4 \in \mathbb{C}$. Then either (a_1, a_2, a_3, a_4) and $(-a_2, a_1, -a_4, a_3)$ are linearly independent (over \mathbb{C}), or $a_2 = -\lambda a_1$ and $a_4 = -\lambda a_3$ for $\lambda = \pm i$.*

Lemma 2.4 is proved by observing that if $a_1 = \lambda(-a_2)$ and $a_2 = \lambda a_1$, then either $\lambda = \pm i$, or $\lambda = 0$ (in which case $a_1 = a_2 = a_3 = a_4 = 0$, so we can again take $\lambda = \pm i$). Define

$$\begin{aligned}\Pi_1 &= \{(a_1, a_2, a_3, a_4) \in \mathbb{C}^4 : a_2 = -ia_1, a_4 = -ia_3\}, \\ \Pi_2 &= \{(a_1, a_2, a_3, a_4) \in \mathbb{C}^4 : a_2 = ia_1, a_4 = ia_3\}.\end{aligned}\tag{6}$$

Note that Π_1 and Π_2 are (two-dimensional) planes in \mathbb{C}^4 . Moreover, for any $(a + bi, -b + ai, c + di, -d + ci) \in \Pi_2$, we have $(a - bi, -b - ai, c - di, -d - ci) \in \Pi_1$. That is, Π_2 is the complex conjugate of Π_1 .

3 Foliations and Frobenius' theorem

Throughout this section we use various concepts from differential geometry, such as tangent bundles and foliations. A reader who is unfamiliar with such concepts can find a nice introduction in [15, Chapter 19] (one may also wish to look at Chapter 8).

We will often refer to the point $0 \in \mathbb{R}^d$. Sometimes, however, the dimension of the underlying vector space may be ambiguous. Where it is helpful, we write 0_d to remind the reader that the point 0 belongs to the vector space \mathbb{R}^d .

A *chart* of a manifold $M \subset \mathbb{R}^d$ is a pair (U, φ) where U is an open subset of M and $\varphi : U \rightarrow \hat{U}$ is a homeomorphism from U to the open subset $\hat{U} = \phi(U) \subset \mathbb{R}^d$. Given two charts $(U_1, \varphi_1), (U_2, \varphi_2)$ of M such that $U_1 \cap U_2 \neq \emptyset$, the *transition map* of φ_1 and φ_2 is $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$. For a given integer $0 < d' < d$ and a chart (U, φ) , a d' -dimensional *plaque* of U is the set of points of U that φ takes into a d' -dimensional hyperplane defined by $x_1 = c_1, \dots, x_{d-d'} = c_{d-d'}$ (for some constants $c_1, \dots, c_{d-d'}$).

A d' -dimensional *foliation* of M is a covering of M by charts (U_j, φ_j) such that for every two charts $(U_j, \varphi_j), (U_k, \varphi_k)$ with $U_j \cap U_k \neq \emptyset$, the d' -dimensional plaques of the two charts overlap. More rigorously, the transition map $\varphi_k \circ \varphi_j^{-1}$ takes every d' -dimensional hyperplane that is defined by $x_1 = c_1, \dots, x_{d-d'} = c_{d-d'}$ to a d' -dimensional hyperplane that is defined by $x'_1 = c'_1, \dots, x'_{d-d'} = c'_{d-d'}$. By piecing together overlapping plaques we obtain a subdivision of M into disjoint connected d' -dimensional injectively immersed submanifolds. These submanifolds are called the *leaves* of the foliation.

Let M be a d -dimensional smooth manifold. The *tangent bundle* TM is a $2d$ -dimensional smooth manifold which is the disjoint union of the tangent spaces $\{T_\zeta\}_{\zeta \in M}$. Each element of the tangent bundle can be identified with a pair (ζ, v) , where $\zeta \in M$ and $v \in T_\zeta M$.

Let $E \subset TM$ be a $(d + d')$ -dimensional sub-manifold of TM . We say that E is a d' -dimensional *sub-bundle* of TM if for every $\zeta \in M$, we have $(\{\zeta\} \times T_\zeta M) \cap E = \{\zeta\} \times V$, where V is a d' -dimensional vector subspace of $T_\zeta M = \mathbb{R}^d$. We will call this subspace $E(\zeta) \subset \mathbb{R}^d$. Intuitively, the vector space $E(\zeta)$ varies smoothly as the base-point ζ changes.

A *vector field* on M is a smooth function $X : M \rightarrow TM$ that assigns an element of $T_\zeta M$ to each point $\zeta \in M$. We will abuse notation slightly and write $X(\zeta) = v$ to mean $X(\zeta) = (\zeta, v) \in TM$. If E is a sub-bundle of TM and $X : M \rightarrow TM$ is a vector field, we say that X *takes values in E* if

$X(\zeta) \in E$ for all $\zeta \in M$. Let $C^\infty(M)$ be the set of smooth functions $M \rightarrow \mathbb{R}$. For each $f \in C^\infty(M)$, define $X(f) \in C^\infty(M)$ to be the function where $X(f)(\zeta)$ is the directional derivative of f at ζ in the direction $X(\zeta)$. Thus, we can think of a vector field as a map $X: C^\infty(M) \rightarrow C^\infty(M)$. Since the domain and range of X is the same space (namely, $C^\infty(M)$), vector fields can be composed.

Given two smooth vector fields $X, Y: M \rightarrow TM$, their *Lie bracket* $[X, Y]$ is the smooth vector field that satisfies

$$[X, Y](f) = X \circ Y(f) - Y \circ X(f) \quad \text{for all } f \in C^\infty(M).$$

See [15] for further details. In particular, the Lie bracket always exists.

A sub-bundle $E \subset TM$ is called *involutive* (or *integrable*) if for every two vector fields X, Y that take values in E , the Lie bracket $[X, Y]$ takes values in E . We say that E *arises from a foliation* of M if there exists a foliation \mathcal{F} of M with the following property: if $\zeta \in M$ and $L \subset M$ is the leaf passing through ζ , then $E(\zeta) = T_\zeta L$. Similarly, we say that the foliation \mathcal{F} corresponds to E .

Theorem 3.1 (Frobenius). *A sub-bundle $E \subset TM$ is involutive if and only if E arises from a foliation of M .*

A proof of Theorem 3.1 can be found, for example, in [15, Chapter 19]. Similarly, the following lemma can be found in [15, Theorem 19.21].

Lemma 3.2. *Let M be a d -dimensional manifold and let \mathcal{F} be a foliation of M into d' -dimensional leaves. Let $E \subset TM$ be the sub-bundle that arises from \mathcal{F} . Let L be a maximal (under set inclusion) connected d' -dimensional sub-manifold of M , such that for every point $\zeta \in L$, $T_\zeta L = E(\zeta)$. Then L is a leaf of the foliation \mathcal{F} .*

We are now ready to show that if Z is a bounded-degree hypersurface in \mathbb{R}^4 , then for a generic point $z \in Z$ there is at most one irreducible curve $\gamma \subset \mathbb{C}^2$ that satisfies $z \in \iota(\gamma) \subset Z$.

Theorem 3.3. *Let $P \in \mathbb{R}[x_1, y_1, x_2, y_2]$ be an irreducible polynomial of degree at most D . Then there exists a variety $Y \subset \mathbb{C}^4$ of dimension at most two and degree $O_D(1)$ such that for every $p \in \mathbf{Z}_\mathbb{R}(P) \setminus Y(\mathbb{R})$, there is at most one irreducible complex curve $\gamma \subset \mathbb{C}^2$ with $p \in \iota(\gamma)_\text{reg}$ and $\iota(\gamma) \subset \mathbf{Z}_\mathbb{R}(P)$.*

Proof. Set $Z = \mathbf{Z}_\mathbb{C}(P)$. If Z_reg does not contain any real points, then the theorem is immediately obtained by setting $Y = Z_\text{sing}$. We may thus assume that Z_reg contains at least one real point z . Since $T_z Z$ is the complexification of the linear subspace $T_z \mathbf{Z}_\mathbb{R}(P) \subset \mathbb{R}^4$, we get that $T_z Z$ is closed under complex conjugation. We recall the planes Π_1 and Π_2 that were defined in (6). If $T_z Z$ contains one of these planes then it also contains the other (since Π_1 is the complex conjugate of Π_2). This is impossible since $T_z Z$ is a hyperplane in \mathbb{C}^4 while the span of Π_1 and Π_2 is \mathbb{C}^4 . We conclude that for each real point $z \in Z$, $T_z Z$ does not contain Π_1 or Π_2 .

Let Y_0 be the set of points $z \in Z_\text{reg}$ such that $T_z Z$ contains Π_1 or Π_2 , and set $Y = Z_\text{sing} \cup Y_0$. By Lemma 2.3, Z_sing is a subvariety of dimension at most two and degree $O_D(1)$. The set Y is also a subvariety of degree $O_D(1)$, since having $T_z Z$ contain Π_j is equivalent to the gradient of P being orthogonal to two vectors that span Π_j (and can thus be described by two equations of degree at most D). By the previous paragraph at least one point of Z_reg is not in Y_0 , so Y cannot be three-dimensional. That is, Y is a subvariety of Z of dimension at most two and degree $O_D(1)$. We set $M = \mathbf{Z}_\mathbb{R}(P) \setminus Y$ and note that M is a three-dimensional manifold in \mathbb{R}^4 .

Let $J: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be a linear transformation that is defined as

$$J(x_1, y_1, x_2, y_2) = (-y_1, x_1, -y_2, x_2).$$

Notice that for any variety $V \subset \mathbb{R}^4$ we have $J(J(V)) = -V$. Thus, for any linear subspace $V \subset \mathbb{R}^4$ we have $J(J(V)) = V$. For every point $p \in M$ we define the linear subspace $E_p = T_p M \cap J^{-1}(T_p M)$. Intuitively, E_p is the largest subset of $T_p M$ that is invariant under J . Since the linear subspaces $T_p M$ and $J^{-1}(T_p M)$ are both three-dimensional and not identical, E_p is a two-dimensional linear subspace. As p varies, the union of the $p \times E_p$ forms a two-dimensional sub-bundle E of the tangent bundle TM .

It can be easily verified that for any two vector fields $\tilde{X}, \tilde{Y} : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ we have $[J(\tilde{X}), J(\tilde{Y})] = J([\tilde{X}, \tilde{Y}])$ (e.g., see [15, Corollary 8.31]). Let $X, Y : M \rightarrow TM$ be vector fields that take values in E and satisfy $J(X(p)), J(Y(p)) \in T_p M$ for each $p \in M$. Since $[J(X), J(Y)] = J([X, Y])$, we get $[J(X), J(Y)](p) \in T_p M$ for each $p \in M$ (e.g., see [15, Corollary 8.32]). By recalling the definition of E_p and since both $[X, Y]$ and $J([X, Y])$ are in $T_p M$, we obtain $[X, Y] \in E_p$ (for every $p \in M$). This in turn implies that E is involutive. Theorem 3.1 then implies that E arises from a foliation of M .

Consider an irreducible complex curve $\gamma \subset \mathbb{C}^2$ and a point $p \in M$ such that $\iota(\gamma) \subset \mathbf{Z}_{\mathbb{R}}(P)$ and $p \in \iota(\gamma)_{\text{reg}}$. Consider a polynomial $f \in \mathbb{C}[z_1, z_2]$ that satisfies $\gamma = Z(f)$, and write $f = u + iv$ with $u, v \in \mathbb{R}[x_1, y_1, x_2, y_2]$. Then $T_p \iota(\gamma)$ is the two dimensional plane that is orthogonal to both $\nabla u(p)$ and $\nabla v(p)$. By the Cauchy-Riemann equations (5), we have that $\nabla u(p) = J(\nabla v(p))$. Thus, $T_p \iota(\gamma)$ is invariant under J . Since $T_p \iota(\gamma) \subset T_p M$, we have $T_p \iota(\gamma) = E_p$. Combining this with Lemma 3.2 implies that every connected component of $\iota(\gamma) \cap M$ is a leaf of the foliation. Let ℓ be the leaf of the foliation that contains p . Then γ is the Zariski closure in \mathbb{C}^2 of $\iota^{-1}(\ell)$. This completes the proof of the theorem, since the complex curve γ is uniquely determined. \square

Corollary 3.4. *Let $P \in \mathbb{R}[x_1, y_1, x_2, y_2]$ be a polynomial of degree at most D . Then there exists a variety $Y \subset \mathbb{C}^4$ of dimension at most two and degree $O_D(1)$ such that for every $p \in \mathbf{Z}_{\mathbb{R}}(P) \setminus Y(\mathbb{R})$, there is at most one irreducible complex curve $\gamma \subset \mathbb{C}^2$ with $p \in \iota(\gamma)_{\text{reg}}$ and $\iota(\gamma) \subset \mathbf{Z}_{\mathbb{R}}(P)$.*

Proof. Let Z_1, \dots, Z_ℓ be the irreducible components of $\mathbf{Z}_{\mathbb{C}}(P)$; note that $\ell \leq D$. For each index j , apply Theorem 3.3 to each variety Z_j , and let Y_j be the corresponding variety from the statement of the theorem. Set $Y = \mathbf{Z}_{\mathbb{C}}(P)_{\text{sing}} \cup (\bigcup_{j=1}^{\ell} Y_j)$. If a point $p \in \mathbf{Z}_{\mathbb{R}}(P)$ is contained in more than one component Z_j then $p \in \mathbf{Z}_{\mathbb{C}}(P)_{\text{sing}}$ and thus $p \in Y$. If p is contained in a single component Z_j then by Theorem 3.3 there is at most one irreducible complex curve $\gamma \subset \mathbb{C}^2$ such that $p \in \iota(\gamma)_{\text{reg}}$ and $\iota(\gamma) \subset Z_j$. \square

4 Proof of Theorem 1.3

We are now ready to prove Theorem 1.3. For the reader's convenience we will restate it here.

Theorem 1.3. *For each $k \geq 1$, $D \geq 1$, $s \geq 1$, and $\epsilon > 0$, there is a constant $C = C_{\epsilon, D, s, k}$ such that the following holds. Let $\mathcal{P} \subset \mathbb{C}^2$ be a set of m points and let \mathcal{C} be a set of n complex algebraic curves of degree at most D . Suppose that $(\mathcal{P}, \mathcal{C})$ has k degrees of freedom and multiplicity type s . Then*

$$I(\mathcal{P}, \mathcal{C}) \leq C \left(m^{\frac{k}{2k-1} + \epsilon} n^{\frac{2k-2}{2k-1}} + m + n \right).$$

Proof. We will make crucial use of the Guth-Katz polynomial partitioning technique from [12, Theorem 4.1].

Theorem 4.1. *Let \mathcal{P} be a set of m points in \mathbb{R}^d . For each $r \geq 1$, there exists a polynomial P of degree at most r such that $\mathbb{R}^d \setminus \mathbf{Z}(P)$ is a union of $O(r^d)$ connected components (cells), and each cell contains $O(m/r^d)$ points of \mathcal{P} .*

Since the curves of \mathcal{C} have k degrees of freedom, the Kővári-Sós-Turán theorem (e.g., see [16, Section 4.5]) implies $I(\mathcal{P}, \mathcal{C}) = O(mn^{1-1/k} + n)$. When $m = O(n^{1/k})$, this implies the bound $I(\mathcal{P}, \mathcal{C}) = O(n)$. Thus, we may assume that

$$n = O\left(m^k\right). \quad (7)$$

We will prove by induction on $m + n$ that

$$I(\mathcal{P}, \mathcal{C}) \leq \alpha_1 m^{\frac{k}{2k-1} + \varepsilon} n^{\frac{2k-2}{2k-1}} + \alpha_2(m + n),$$

where α_1, α_2 are sufficiently large constants. The base case where $m + n$ is small can be handled by choosing sufficiently large values of α_1 and α_2 . In practice, we will bound $I(\iota(\mathcal{P}), \iota(\mathcal{C}))$. Since $\iota: \mathbb{C}^2 \rightarrow \mathbb{R}^4$ is a bijection, $I(\mathcal{P}, \mathcal{C}) = I(\iota(\mathcal{P}), \iota(\mathcal{C}))$.

Partitioning \mathbb{R}^4 . Let P be a partitioning polynomial of degree at most r , as described in Theorem 4.1. The constant r is taken to be sufficiently large, as described below. The asymptotic relations between the various constants in the proof are

$$2^{1/\varepsilon} \ll r \ll \alpha_2 \ll \alpha_1.$$

Let $\Omega_1, \dots, \Omega_\ell$ be the cells of the partition; we have $\ell = O(r^4)$. Let \mathcal{V}_i be the set of varieties from $\iota(\mathcal{C})$ that intersect the interior of Ω_i and let \mathcal{P}_i be the set of points $p \in \mathcal{P}$ such that $\iota(p) \in \Omega_i$. Let $m_j = |\mathcal{P}_j|$, $m' = \sum_{j=1}^\ell m_j$, and $n_j = |\mathcal{V}_j|$. By Theorem 4.1, $m_j = O(m/r^4)$ for every $1 \leq j \leq \ell$.

By [24, Theorem A.2], every variety from \mathcal{V} intersects $O(r^2)$ cells of $\mathbb{R}^4 \setminus \mathbf{Z}(P)$. Therefore, $\sum_{j=1}^\ell n_j = O(nr^2)$. Combining this with Hölder's inequality implies

$$\sum_{j=1}^\ell n_j^{\frac{2k-2}{2k-1}} = O\left((nr^2)^{\frac{2k-2}{2k-1}} \ell^{\frac{1}{2k-1}}\right) = O\left(n^{\frac{2k-2}{2k-1}} r^{\frac{4k}{2k-1}}\right).$$

By the induction hypothesis, we have

$$\begin{aligned} \sum_{j=1}^\ell I(\mathcal{P}_j, \mathcal{V}_j) &\leq \sum_{j=1}^\ell \left(\alpha_1 m_j^{\frac{k}{2k-1} + \varepsilon} n_j^{\frac{2k-2}{2k-1}} + \alpha_2(m_j + n_j) \right) \\ &\leq O\left(\alpha_1 m^{\frac{k}{2k-1} + \varepsilon} r^{-\frac{4k}{2k-1} - 4\varepsilon} \sum_{j=1}^\ell n_j^{\frac{2k-2}{2k-1}} \right) + \sum_{j=1}^\ell \alpha_2(m_j + n_j) \\ &\leq O\left(\alpha_1 r^{-\varepsilon} m^{\frac{k}{2k-1} + \varepsilon} n^{\frac{2k-2}{2k-1}} \right) + \alpha_2(m' + O(nr^2)). \end{aligned}$$

By (7), we have $n^{\frac{1}{2k-1}} = O\left(m^{\frac{k}{2k-1}}\right)$, which in turn implies $n = O\left(m^{\frac{k}{2k-1}} n^{\frac{2k-2}{2k-1}}\right)$. Thus, when α_1 is sufficiently large with respect to r and α_2 , we have

$$\sum_{j=1}^\ell I(\mathcal{P}_j, \mathcal{V}_j) = O\left(\alpha_1 r^{-\varepsilon} m^{\frac{k}{2k-1} + \varepsilon} n^{\frac{k}{2k-1}} \right) + \alpha_2 m'.$$

By taking r to be sufficiently large with respect to ε and the implicit constant in the O -notation, we have

$$\sum_{j=1}^\ell I(\mathcal{P}_j, \mathcal{V}_j) \leq \frac{\alpha_1}{2} m^{\frac{k}{2k-1} + \varepsilon} n^{\frac{2k-2}{2k-1}} + \alpha_2 m',$$

i.e.,

$$I(\iota(\mathcal{P}) \setminus \mathbf{Z}_{\mathbb{R}}(P), \iota(\mathcal{C})) \leq \frac{\alpha_1}{2} m^{\frac{k}{2k-1} + \varepsilon} n^{\frac{2k-2}{2k-1}} + \alpha_2 m'. \quad (8)$$

Incidences on the partitioning hypersurface. It remains to bound incidences with points that are on the partitioning hypersurface $\mathbf{Z}(P)$. To do this, we will make use of the point-curve bound from Theorem 1.2.

Lemma 4.2. *Let $\mathcal{P} \subset \mathbb{C}^2$. Let \mathcal{C} be a set of complex curves of degree at most C_0 such that $(\mathcal{P}, \mathcal{C})$ has k degrees of freedom and multiplicity type s . Let $Y \subset \mathbb{C}^4$ be an algebraic variety of degree at most C_1 . Suppose that for each $\gamma \in \mathcal{C}$, the intersection $\iota(\gamma) \cap Y(\mathbb{R})$ is a real algebraic variety of dimension at most one. Then*

$$I(\iota(\mathcal{P}) \cap Y(\mathbb{R}), \iota(\mathcal{C})) = O(|\mathcal{P}|^{k/(2k-1)} |\mathcal{C}|^{(2k-2)/(2k-1)} + |\mathcal{P}| + |\mathcal{C}|), \quad (9)$$

where the implicit constant depends on k , s , C_0 , and C_1 .

Proof. Let $\pi: \mathbb{R}^4 \rightarrow \mathbb{R}^2$ be a generic linear transformation (see Section 2.1). Then for each $\gamma \in \mathcal{C}$, $\pi(\iota(\gamma) \cap Y(\mathbb{R})) \subset \mathbb{R}^2$ is the zero set of a non-zero polynomial of degree $O_{C_0, C_1}(1)$ (e.g., see [24, Section 5.1]); each set of this form is a union of plane curves and a finite set of points.

Let $\Gamma = \{\pi(\iota(\gamma) \cap Y(\mathbb{R})) : \gamma \in \mathcal{C}\}$. Then Γ is a finite set of (not necessarily irreducible) plane algebraic curves and isolated points, and $(\pi(\iota(\mathcal{P})), \Gamma)$ has k degrees of freedom and multiplicity type $O_{s, C_0, C_1}(1)$.

By Theorem 1.2,

$$I(\pi(\iota(\mathcal{P})), \Gamma) = O(|\mathcal{P}|^{k/(2k-1)} |\mathcal{C}|^{(2k-2)/(2k-1)} + |\mathcal{P}| + |\mathcal{C}|), \quad (10)$$

where the implicit constant depends on k , s , C_0 , and C_1 . Since each incidence in $I(\iota(\mathcal{P}) \cap Y(\mathbb{R}), \iota(\mathcal{C}))$ appears as an incidence in $I(\pi(\iota(\mathcal{P})), \Gamma)$, (10) implies (9). \square

We are now ready to bound the number of incidences involving points lying on $\mathbf{Z}_{\mathbb{R}}(P)$. Let $\mathcal{P}_0 = \iota(\mathcal{P}) \cap \mathbf{Z}_{\mathbb{R}}(P)$, let $m_0 = |\mathcal{P}_0| = m - m'$, and let $\mathcal{C}_0 = \{\gamma \in \mathcal{C} : \iota(\gamma) \subset \mathbf{Z}_{\mathbb{R}}(P)\}$. By Lemma 2.3, for each $\gamma \in \mathcal{C}$, we have that $\iota(\gamma)_{\text{sing}} = \iota(\gamma_{\text{sing}})$ is a finite set of size $O_D(1)$, hence

$$|\{(p, \gamma) \in \mathcal{P}_0 \times \mathcal{C} : \iota(p) \in \iota(\gamma)_{\text{sing}}\}| = O_D(n). \quad (11)$$

Apply Corollary 3.4 to P , and let $Y \subset \mathbb{C}^4$ be the resulting variety. Then

$$|\{(p, \gamma) \in \mathcal{P}_0 \times \mathcal{C}_0 : \iota(p) \in \mathbf{Z}_{\mathbb{R}}(P) \setminus Y(\mathbb{R}), \iota(p) \in \iota(\gamma)_{\text{reg}}\}| \leq m_0. \quad (12)$$

Since Y is an algebraic variety of degree $O_r(1)$ and dimension at most two, at most $O_r(1)$ varieties of the form $\iota(\gamma)$ can be contained in $Y(\mathbb{R})$. Thus

$$|\{(p, \gamma) \in \mathcal{P}_0 \times \mathcal{C}_0 : \iota(p) \in \iota(\gamma)_{\text{reg}}, \iota(\gamma) \subset Y(\mathbb{R})\}| = O_r(m_0). \quad (13)$$

Let $\mathcal{C}' = \mathcal{C} \setminus \mathcal{C}_0$. It remains to control the size of the sets

$$\{(p, \gamma) \in \mathcal{P}_0 \times \mathcal{C}' : \iota(p) \in \iota(\gamma)_{\text{reg}}\}$$

and

$$\{(p, \gamma) \in \mathcal{P}_0 \times \mathcal{C}_0 : \iota(\gamma) \not\subset Y(\mathbb{R}), \iota(p) \in (\iota(\gamma)_{\text{reg}} \cap Y(\mathbb{R}))\}.$$

By Lemma 4.2, both of these sets have size

$$O(m_0^{k/(2k-1)} n^{(2k-2)/(2k-1)} + m_0 + n). \quad (14)$$

Combining (11), (12), (13), and (14) implies

$$I(\mathcal{P}_0, \iota(\mathcal{C})) = O(m_0^{k/(2k-1)} n^{(2k-2)/(2k-1)} + m_0 + n).$$

Taking α_1, α_2 to be sufficiently large with respect to the constant of the O -notation, we have

$$I(\iota(\mathcal{P}) \cap \mathbf{Z}_{\mathbb{R}}(P), \iota(\mathcal{C})) \leq \frac{\alpha_1}{2} m^{k/(2k-1)} n^{(2k-2)/(2k-1)} + \alpha_2(m_0 + n). \quad (15)$$

Combining (15) and (8) completes the induction. \square

Acknowledgements. The authors would like to thank Orit Raz and Frank de Zeeuw for a discussion that pushed us to work on this problem, and we would like to thank the anonymous referee for numerous suggestions and recommendations. Part of this research was performed while the authors were visiting the Institute for Pure and Applied Mathematics (IPAM) in Los Angeles, which is supported by the National Science Foundation. The second author was supported by NKFIH K115799. The third author was supported in part by an NSF Postdoctoral Fellowship.

References

- [1] J. Bochnak, M. Coste, and M.-F. Roy. *Real algebraic geometry*. Springer-Verlag, Berlin, 1998.
- [2] J. Bourgain and C. Demeter. l^p decouplings for hypersurfaces with nonzero Gaussian curvature. *arXiv:1407.0291*, 2014.
- [3] J. Bourgain and C. Demeter. New bounds for the discrete Fourier restriction to the sphere in four and five dimensions. *Internat. Math. Res. Notices*, pages 3150–3184, 2015.
- [4] E. Breuillard, B. Green, and T. Tao. Approximate subgroups of linear groups. *Geom. Funct. Anal.*, 21(4):774–819, 2011.
- [5] K. L. Clarkson, H. Edelsbrunner, L. J. Guibas, M. Sharir, and E. Welzl. Combinatorial complexity bounds for arrangements of curves and spheres. *Discrete Comput. Geom.*, 5(2):99–160, 1990.
- [6] D. Cox, J. Little, and D. O’Shea. *Ideals, varieties, and algorithms*. Springer, New York, third edition, 2007.
- [7] T. W. Dubé. The structure of polynomial ideals and Gröbner bases. *SIAM J. Comput.*, 19(4):750–775, 1990.
- [8] Z. Dvir and S. Gopi. On the number of rich lines in truly high dimensional sets. *Proc. of 31st International Symposium on Computational Geometry*, pages 584–598, 2015.
- [9] G. Elekes and E. Szabó. How to find groups?(and how to use them in erdős geometry?). *Combinatorica*, 32(5):537–571, 2012.
- [10] M. Farber, S. Ray, and S. Smorodinsky. On totally positive matrices and geometric incidences. *J. Combin. Theory Ser. A*, 128:149–161, 2014.

- [11] P. Gianni, B. Trager, and G. Zacharias. Gröbner bases and primary decomposition of polynomial ideals. *J. Symbolic Comput.*, 6(2-3):149–167, 1988. Computational aspects of commutative algebra.
- [12] L. Guth and N. Katz. On the Erdős distinct distance problem in the plane. *Ann. of Math.*, 181:155–190, 2015.
- [13] J. Harris. *Algebraic geometry: A first course*, volume 133 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
- [14] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977.
- [15] J. M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2003.
- [16] J. Matoušek. *Lectures on discrete geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.
- [17] D. Mumford. *Algebraic geometry. I*. Springer-Verlag, Berlin-New York, 1981. Complex projective varieties, Corrected reprint.
- [18] J. Pach and M. Sharir. Repeated angles in the plane and related problems. *J. Combin. Theory Ser. A*, 59(1):12–22, 1992.
- [19] J. Pach and M. Sharir. On the number of incidences between points and curves. *Combin. Probab. Comput.*, 7(1):121–127, 1998.
- [20] O. Raz, M. Sharir, and F. de Zeeuw. Polynomials vanishing on cartesian products: The Elekes-Szabó theorem revisited. *Proc. 31st Symp. on Comp. Geom.*, pages 522–536, 2015.
- [21] O. Raz, M. Sharir, and J. Solymosi. Polynomials vanishing on grids: The Elekes-Rónyai problem revisited. *Amer. J. Math.*, to appear, 2014.
- [22] M. Sharir and J. Zahl. Cutting algebraic curves into pseudo-segments and applications. *arXiv:1604.07877*, 2016.
- [23] J. Solymosi and F. de Zeeuw. Incidence bounds for complex algebraic curves on Cartesian products. *New Trends in Intuitive Geometry*, to appear, 2016.
- [24] J. Solymosi and T. Tao. An incidence theorem in higher dimensions. *Discrete Comput. Geom.*, 48(2):255–280, 2012.
- [25] E. Szemerédi and W. T. Trotter, Jr. Extremal problems in discrete geometry. *Combinatorica*, 3(3-4):381–392, 1983.
- [26] C. Tóth. The Szemerédi-Trotter theorem in the complex plane. *Combinatorica*, 35(1):95–126, 2015.
- [27] C. Valculescu and F. de Zeeuw. Distinct values of bilinear forms on algebraic curves. *arXiv:1403.3867*, 2014.
- [28] J. Zahl. A Szemerédi-Trotter type theorem in \mathbb{R}^4 . *Discrete Comput. Geom.*, 54(3):513–572, 2015.
- [29] J. Zahl. A note on rich lines in truly high dimensional sets. *Forum Math. Sigma*, (4):1–13, 2016.